Ross Program 2006 Application Problems

This document is part of the application to the Ross Mathematics Program, and is posted at: www.math.ohio-state.edu/ross. This excellent eight-week residential program for high school students will run from June 19 to August 11, 2006. The application deadline is June 1, but we recommend that you send in your application some time earlier.

Applicants should work on each of the 9 problems below.

- *No one is expected to solve all the problems.*

We are interested in seeing how you approach an unfamiliar math problem, not whether you can find the answers by searching.

> Note: Ross Program courses do NOT involve a wide variety of problems like these.
> Instead each course concentrates deeply on one subject.
> This Application problem set is our attempt to assess your general mathematical background and interests.

For each problem, explore the situation (with tables, with pictures, etc), make some guesses, test the truth of those conjectures, and describe the progress you have made.

Where were you led by your experimenting?

Include your thoughts even though you may not have completely solved the problem. If you've seen one of the problems before (e.g. in a class or in a book), please include a reference with your solution.

Write up your problem solutions on separate pages (one side only) in black ink or dark pencil. Send them, along with your Application Form, to the following address.

Ross Mathematics Program
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The Ohio State University
231 W. 18th Ave.
Columbus, OH 43210
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Electronic submissions are discouraged.

(1) The letters $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ represent seven positive whole numbers. The letters $b_1, b_2, b_3, b_4, b_5, b_6, b_7$ represent the same numbers but in a different order. Will the value of the product

$$(a_1 - b_1)(a_2 - b_2)(a_3 - b_3)(a_4 - b_4)(a_5 - b_5)(a_6 - b_6)(a_7 - b_7)$$

always be an even number? Explain your conclusion.

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(2) Call a number "nice" if it can be expressed as a sum of two or more consecutive positive integers. For example, 5 and 6 are nice numbers because $5 = 2+3$ and $6 = 1+2+3$.

(a) Which numbers from 1 to 50 are not nice?
What's the pattern of nice numbers for sizes beyond 50?

(b) Explain why the pattern you observed holds true generally.

(c) Some numbers are "very nice", in the sense that they are nice in more than one way. For example, 15 is very nice because $15 = 1+2+3+4+5 = 4+5+6 = 7+8$.
Is 1000 a very nice number?
How can you detect whether a given number is very nice? Explain your answer.

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(3) Twelve students use a dining room which has three tables with four place settings on each table. How should one vary the seating arrangement so that in five days, each student shares the table at dinner with each other student?

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(4) A set of numbers has "the triple-sum property" (or TSP) if there are three numbers in the set whose sum is also in the set. [Repetitions are allowed.]
For example, the set $U = \{2, 3, 7\}$ has TSP since $2 + 2 + 3 = 7$, while $V = \{2, 3, 10\}$ fails to have TSP.

(a) Suppose the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ is separated into two parts, forming two subsets A and B.

<u>Prove</u>: Either A or B must have the triple-sum property.

To begin the proof, suppose that statement is false and there are sets A and B as above, each without TSP. If 1 lies in A then $3 = 1 + 1 + 1$ must be in B. Complete the proof that this situation is impossible.

(b) Is a similar result true when the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is separated into two parts?

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(5) Suppose 100 dots are arranged in a square 10×10 array, and each dot is colored red or blue.

(a) Prove that this array must contain a "monochromatic" rectangle. That is, no matter how the red and blue colors are assigned, there must be either a set of four red dots that form a rectangle or else a set of four blue dots that form a rectangle.

Note: We aren't considering the colors of the dots inside that rectangle. Just the corner points.
Let's consider only those rectangles having horizontal and vertical sides.

(b) Does this result remain true for smaller rectangular arrays of dots?
For example, must a monochromatic rectangle exist in a 5×5 array? In a 4×6 array?

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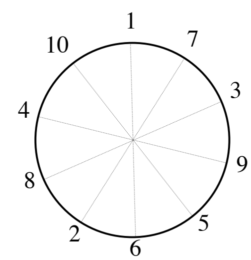
(6) A <u>balanced *n*-wheel</u> is a placement of the numbers $1, 2, 3, \ldots, 2n$ around a circle, in such a way that the sum of any two adjacent numbers equals the sum of the two numbers on the opposite side of the circle.

Here's an example of a balanced 5-wheel:
Check: 1+7 = 6+2; 7+3 = 2+8; 3+9 = 8+4, etc.



Is there a balanced 3-wheel?
Is there a balanced 4-wheel?
For which n does a balanced n-wheel exist?

Justify your answers.

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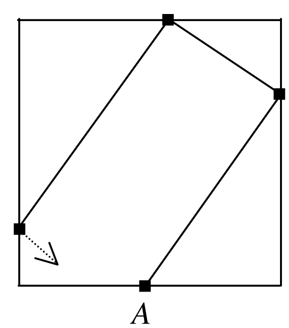
(7) Robot Robbie starts at a point A on the edge of a large square and travels into the square in a certain direction. Robbie travels in straight lines inside the square, but when reaching an edge he turns 90° left and moves in that new direction. If he is blocked by an edge after the left turn, he tries a right turn. If both turns are blocked he turns around and retraces the previous path.

(a) For certain initial directions his path will return to the point A after three turns. How many such angles are there?
(Can you prove your answer is correct?)

(b) If Robbie does not return to A after three turns, could he return to A at some future time?



(c) What sort of path does Robbie follow? Will he bounce around randomly in the square or will his path eventually stabilize in some way?

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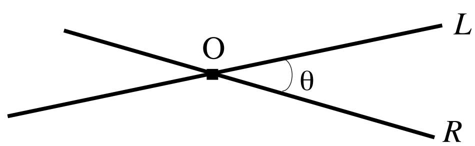
(8) What numbers can be expressed as an alternating-sum of some increasing sequence of distinct powers of 2 ? To form such a sum, choose a subset of the sequence 1, 2, 4, 8, 16, 32, 64, . . . (these are the powers of 2). List the numbers in that subset in increasing order and combine them with alternating plus and minus signs. For example, $1 = -1 + 2$; $2 = -2 + 4$; $3 = 1 - 2 + 4$; $4 = -4 + 8$; $5 = 1 - 4 + 8$; $6 = -2 + 8$; etc.

(a) Is every positive integer expressible in this fashion? If so, give a convincing proof.

(b) There can be more than one expression of this type for a given number. For instance $5 = 1 - 4 + 8$ and $5 = -1 + 2 - 4 + 8$. Given a number n, how many different ways are there to write n in this way? Explain why your answer is correct.

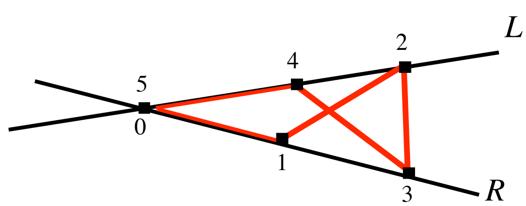
(9) Suppose lines L and R intersect at point O, and let θ be the measure of that angle.



Starting at O, take steps alternately on the lines L and R never stepping back to the spot just used. All the steps have the same length.

(a) For a certain angle θ (as in the sketch) we arrive back at O after exactly five steps. How big is θ in that case?

(Sketch isn't drawn to scale.)



(b) For what angle θ do we arrive back at O after six steps?
What about seven steps?

(10) Which of the problems here did you like the best? Why?

Good luck with the problems!

Further information about this summer mathematics program is available on the web at www.math.ohio-state.edu/ross or by email at `ross@math.ohio-state.edu`.